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document Quasiperiodic waves at the onset of zero Prandtl number convection with rotation Krishna Kumar[1], Sanjay Chaudhuri[2], and Alaka Das[1] [1]Physics and Applied Mathematics Unit, [2]Statistics and Mathematics Unit

abstract We show the possibility of quasiperiodic waves at the onset of thermal convection in a thin horizontal layer of slowly rotating zero-Prandtl number Boussinesq fluid confined between stress-free conducting boundaries. Two independent frequencies emerge due to an interaction between a stationary instability and a self-tuned wavy instability in presence of coriolis force, if Taylor number is raised above a critical value. Constructing a dynamical system for the hydrodynamical problem, the competition between the interacting instabilities is analyzed. The forward bifurcation from the conductive state is self-tuned.

Thermal convection in Boussinesq fluids in the limit of vanishing Prandtl number $P(=\nu/\kappa)$ [1-12] is of interest for astrophysical problems ($P \approx 10^{-8}$) as well as for liquid metals ($P \approx 10^{-2} - 10^{-3}$). The theoretical study, in particular with *stress-free* boundary conditions, in the limit of large thermal diffusivity κ has been considered subtle for a long time because the linearly unstable two-dimensional (2D) rolls become exact nonlinear solution. The nonlinearity $\mathbf{v}\cdot\nabla\theta$ due to the advection of temperature fluctuation θ by the velocity field $\mathbf{v}$ might be negligible spiegel. The nonlinearity $\mathbf{v}\cdot\nabla\mathbf{v}$, due to the self interaction of velocity field, does not contribute to saturation for straight (2D) rolls. This led to the speculation that the zero P limit might involve a singular limit problem similar to the one with infinite Reynold number in incompressible fluid dynamics. However, the recent three-dimensional (3D) direct numerical simulation (DNS) of zero-Prandtl number Boussinesq equations by Thual thual showed the saturation instead of indefinite growth of the solution even with *stress-free* boundary conditions. He also compared the results of zero P equations with that of the full Oberbeck-Boussinesq equations in the asymptotic limit of vanishing P, and found complete agreement in two cases. The saturation of growing 2D rolls at the onset of convection occurs by generation of *self-tuned* 3D waves, the mechanism of which was explained in a simple model kft. The results of this model, in its validity range, agreed well with that of DNS just above the convective instability. The new nonlocal instability at the onset occurs purely due to nonlinear effects, while linear equations predict the stationary instability chandra.

We present, in this article, a dynamical system constructed for thermal convection in zero Prandtl number Boussinesq fluid, confined between *stress-free* conducting flat boundaries, and subjected to a slow rotation about the vertical axis. We then investigate numerically the system to study the effect of coriolis force on the onset of convection. We show that the convection sets in as quasiperiodic waves at the onset of convective instability for Taylor number T above a critical value T_c, although the *principle of exchange of stability* is valid according to the linear theory chandra for these Taylor numbers. The generation of two independent frequencies is the result of an interaction between a stationary instability and a *self-tuned* wavy perturbations in presence of coriolis force. This an example of a new *self-tuned* forward bifurcation. For the values of Taylor number below T_c, the convection sets in as *self-tuned* wavy instability as is the case in absence of rotation. However, the model shows a possibility transition from one wavy instability to another through a narrow window of period-doubling instability.

We consider a thin layer of a Boussinesq fluid of infinite horizontal extension subjected to a uniform adverse temperature gradient β across the fluid layer, and a rigid body rotation with an angular velocity Ω about the vertical axis. The fluid is assumed to have uniform values of the kinematic viscosity ν and the thermal difusitivity κ. The basic state is the conductive state with no fluid motion in the rotating frame of reference. The convective flow, in the limit of zero-Prandtl number, is then described by the following system of dimesionless hydrodynamic equations, eqnarray $\partial_t(\nabla^2 v_3) = \nabla^4 v_3 + R\nabla_H^2\theta - \sqrt{T}\partial_z\omega_3$

The hydrodynamical equations (vel-temp) are the same as those derived by Chandrasekhar chandra. We have nondimesionlized them and considered the case of zero P. We have also eleminited the pressure term from Navier-Stokes equations by taking curl twice and using the incompressibility condition ($\nabla\cdot\mathbf{v} = 0$). The conclusions derived from the linearized version of the equations remain unchanged even in the present case. Following the arguments of Chandrasekhar chandra, one arrives at the conclusion that the *principle of exchange of stability* is valid even in the limit discussed here. The critical value of Rayleigh number now reads $R_c(T) = \pi^4 x\left[(1+x)^3 + T\pi^4\right]$ and critical wave number $k_c(T) = \pi\sqrt{(l_1 + l_2 - 12)}$ now depend on Taylor number T chandra. In the above, $l_{1,2} = \left[14\left\{12 + T\pi^4 \pm \sqrt{(12+T\pi^4) - 14}\right\}\right]^{13}$ and $x = k_c^2/\pi^2$. In absence of rotation $k_c^0 \equiv k_c(T=0) = \pi/\sqrt{2}$ and $R_c^0 \equiv R_c(T=0) = 27\pi^4/4$.

The 2D rolls are not exact solutions of nonlinear hydrodynamic system with rotation as is the case in zero-Prandtl number convection in absence of rotation. Nevertheless, the growing 2D rolls are not saturated just above the onset of convective instability. The saturation occurs only because of nonlinear interaction of 2D rolls with 3D wavy perturbations. To understand the nonlinear behavior close to the convective instability, we construct a consistent minimal-mode model using Galerkin technique mm. We expand the vertical velocity v_3 and the vertical vorticity ω_3 in Fourier series compatible with the *stress-free* boundary conditions and conducting thermal boundary conditions. As the DNS, in absence of rotation, showed standing patterns kft instead of traveling patterns, we expect similar behavior at least for small rotation rates. Therefore, we expand the fields with real Fourier coefficients. This lead to the following expansion for the vertical velocity and the vertical vorticity in a minimum-mode model. eqnarray $\mathrm{v}_3(x,y,z,t) = W_{101}(t)\cos k_c x \sin \pi z$

The mode selection is quite systematic. As rotation couples the vertical velocity and the vertical vorticity linearly, we have selected the mode ζ_{101}. The mode ζ_{010} is essential to saturate zero Prandtl number convection via wavy instability. All other modes appear through the nonlinear interaction of these vorticity modes with the critical velocity mode W_{101}. As the vorticity field is very crucial for saturation in the limit of vanishing Prandtl number, all relevant second harmonics are retained for vertical vorticity. All relevant harmonics of the vertical velocity field, consistent with the selection of the vertical vorticity, are also retained. Other higher order modes may be required as Rayleigh number is raised further. As we are interested to capture essential nonlinear interaction between competing instabilities just above the onset of convection, these modes are essential. The solenoidal character of the velocity and the vorticity fields yield horizontal components of the velocity and the vorticity fields. The thermal fluctuation θ is captured from Eq. temp. Projecting the hydrodynamic equations (vel - temp) on above modes, we arrive at a twelve-dimensional dynamical system sanjay.

We now investigate the solutions of the dynamical system by performing numerical integration of the model using standard fourth order Runge-Kutta as well as Bulirsh-Stoer schemes. By choosing a value for T, we set $k_c(T)$. We then choose a value for q. We have tried with different values of the wavenumber q of the perturbations and got qualitatively similar results except when the ratio q/k_c is close to unity. We present here all the results for the case $q/k_c(T) = 0.4$. Initial values for all the twelve modes are chosen randomly, and integration is done for a fixed value of Rayleigh number R. We then repeat the process by increasing the value of R in small steps. We have also tried various initial conditions. The results of all the numerical integrations remain the same for the same values of all the relevant parameters. In absence of rotation ($T = 0$), only six modes are excited. This model then reproduces the results of the model kft of zero P convection without rotation. In presence of rotation all twelve modes are excited as it should in a consistent model.

Figure 1 gives the stability boundaries of various possible solutions, in the parameter space $R - T$, computed from the model dynamical system. The lowest line in Fig. 1 shows linear dependence chandra of R_c, the critical Rayleigh number, on Taylor number T for the onset of stationary convection in zero Prandtl number Boussinesq fluid. The onset of overstability for the case of vanishing P for the Taylor numbers considered here is much above, and is not shown in the figure.

As Rayleigh number is raised above its critical value $R_c(T)$ for various values of Taylor number below $T = 6.0$, conduction state becomes unstable via stationary bifurcation. However, 2D rolls with broken mirror symmetry veronis does not saturate until wavy perturbations interact with them. This saturates the growing rolls at finite amplitude even at the onset. The wavy perturbations are automatically generated when the amplitude of the 2D roll mode becomes large enough. The *self-tuned* waves consume the energy of 2D rolls and stop the unbounded growth of latter. This is precisely what happens when there is no rotation and 2D rolls have mirror symmetry.

The three rows of Fig. 2 show projections of the phase diagram, starting from left, in $\zeta_{101} - W_{101}$, $\zeta_{010} - W_{101}$, and $W_{111} - W_{101}$ planes for various Rayleigh numbers (increasing downward) for fixed value of T and q.

As Rayleigh number is increased slowly, the solution changes from one wavy solution to another through a thin regime showing period doubling solutions (see the middle row of Fig. 2). The first wavy solution $SW1$ has 2D mode W_{101} with non-zero mean as in the absence of rotation in zero P convection kft, while the second wavy solution $SW2$ has 2D mode with zero mean. As Rayleigh number is increased, the exchange of energy from 2D modes to waves increases. The larger amplitude variation of vorticity modes is at the cost of energy of 2D rolls. This is well known feature in the case of oscillatory instability.

As T is raised further, the rotation facilitates easily the exchange of more energy from the 2D roll mode W_{1010} to the vertical vorticity mode ζ_{101} through linear coupling. We observe an interesting behavior for $T > 6$ (see Fig. 1). The conduction state becomes unstable via stationary instability chandra but the final state just above onset is quasiperiodic waves db. Figure 3 shows the variation of various modes with time. The amplitudes of all the modes begin modulating at the same frequency. However, the ratio of the frequency of wavy motion and that of amplitude modulation not an integer. The Fourier transform of these modes shows two independent frequencies. The frequency of amplitude modulation is much smaller compared to that for wavy motion. The sharp decrease of the amplitudes of higher order modes confirms the fast convergence of the expansion. The model, therefore, represents accurately the scenario close to the instability onset. Figure 4 shows the projections of phase space trajectories in various planes. It clearly describes the quasiperiodicity of the convective flow. The trajectories are confined in twelve dimensional torous in the phase space. The quasiperiodic behavior originates due to the nonlinear interaction among the 2D velocity mode W_{101}, the 2D vorticity mode ζ_{101} excited by rotation, and the wavy vorticity mode ζ_{010}. Figure 5 reveals some interesting details of the time dependence of of convective patterns. The complex textures of the quasiperiodic patterns are shown over a period of wavy motion, which is much faster than the amplitude modulation. The halves of a period of wavy motion are quite asymmetric. The textures of the pattern at different times are never the same due to quasiperiodicity.

We have presented in this work a simple dynamical system, which describes the phenomenon of thermal convection in rotating Boussinesq fluid of zero Prandtl number very close to the onset. For Taylor number above a critical value T_c, quasiperiodic waves are observed at the instability onset. For very values of Taylor number below T_c, the coriolis force causes one wavy instability to another through period doubling instability. We have shown that convection might be possible as quasiperiodic waves, even if the *principle of exchange of stability* is valid according to linearized hydrodynamical system. The saturation to quasi-periodic convectie state is *self-tuned* purely due to the nonlinear effects. This is an example of new *self-tuned* bifurcation scenario from the conduction state to unsteady convective state. The model presented would also be useful to study an interesting possibility of transition from a state of rest to quasi-periodic chaos rt at the primary instability.

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